JOURNAL OF APPROXIMATION THEORY **78**, 276–285 (1994)

Interpolation of Entire Functions Associated with Some Freud Weights, II

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Communicated by Doron S. Lubinsky

Received September 28, 1992; accepted in revised form June 25, 1993

In [*J. Approx Theory* **71** (1992), 123–137], Al-Jarrah and Hasan considered the weight function $W(x) = \exp(-2|x|^{\alpha})$, $\alpha > 0$, $x \in \Re$, and investigated the growth of an entire function f that guarantees the geometric convergence of the Lagrange and Hermite interpolation processes and the Gauss–Jacobi quadrature formula for f and its higher derivatives when the nodes of interpolation are chosen to be the zeros of the orthogonal polynomial associated with W. In this paper, we repeat investigations similar to those of Al-Jarrah and Hasan, but this time with a more general Freud-type weight function $\hat{W}^2(x) = w^2(x)\exp(-2Q(x))$, $x \in \Re$, where, for example, $w(x)$ is a generalized Jacobi factor, and $Q(x)$ is an even function that satisfies various restrictions.

1. Introduction and Notations

Let $W(x)$ be a weight function on $\Re$, and $\{P_n(W; x)\}_{n=0}^{\infty}$ be the sequence of orthogonal polynomials associated with W. Denote the leading coefficient of $P_n(W)$ by $\gamma_n(> 0)$, and the zeros of $P_n(W)$ by $\{x_{kn}\}_{k=1}^{n}$ with $x_{1n} > x_{2n} > \cdots > x_{nn}$. We also denote, as usual, the Lagrange interpolation polynomial, the Hermite interpolation polynomial, and the Gauss–Jacobi quadrature formula to a function f over the nodes $\{x_{kn}\}_{k=1}^{n}$ by $L_n(W; f)$, $H_n(W; f)$, and $Q_n(W; f)$, respectively. It is known that if f is

0021-9045/94 $6.00

an entire function and $\xi \in \mathscr{C}$, then (see [2, III, Sect. 8.4] and [3])

$$f(\xi) - L_n(W; f; \xi) = \frac{P_n(W; \xi)}{2\pi i} \oint_{C_n} \frac{f(z)\,dz}{P_n(W; z)(z - \xi)}, \tag{1.1}$$

$$f(\xi) - H_n(W; f; \xi) = \frac{P_n^2(W; \xi)}{2\pi i} \oint_{C_n} \frac{f(z)\,dz}{P_n^2(W; z)(z - \xi)}, \tag{1.2}$$

and

$$\int_R f(x) W(x)\,dx - Q_n(W; f) = \frac{1}{2\pi i} \sum_{k=n}^{\infty} \frac{\gamma_{k+1}}{\gamma_k} \oint_{C_k} \frac{f(z)\,dz}{P_k(W; z) P_{k+1}(W; z)}, \tag{1.3}$$

where C_k is the boundary of a simply connected domain containing the zeros of $P_k(W)$ in its interior.

Throughout the rest of this paper, we shall adopt the following assumptions and notation:

A_1: f will be an entire function, and $M_f(R) = \max_{|z|=R} |f(z)|$, $z \in \mathscr{C}$,

A_2: for $\alpha > 0$, $\sigma = \limsup_{R\to\infty} (\log M_f(R))/2R^\alpha$, $\beta_\alpha = \lambda_\alpha^{-1/\alpha}$, and $\lambda_\alpha = \Gamma(\alpha)/(2^{\alpha-2}\Gamma(\alpha/2)^2)$; and

A_3: a is the positive solution of the equation

$$g(x) = \frac{1-x}{4} \exp\left(\frac{1-x}{2x}\right) = 1, \qquad (a \approx 0.23).$$

The following theorem gives a summary of the main results that were proved in [1].

THEOREM A. *Let* $W_\alpha(x) = \exp(-2|x|^\alpha)$, $\alpha > 0$, $x \in \mathfrak{R}$, *and*

$$\tau(\alpha) = \frac{(1-a)^{((\alpha+2)/2)}}{2a(\alpha+2)\beta_\alpha^\alpha}.$$

Then for $m = 0, 1, 2, \ldots,$ *and all* $\xi \in \mathscr{C}$, *we have*

(i) $\sigma < \tau(\alpha) \Rightarrow \limsup_{n\to\infty} \left| \int_R f^{(m)}(x) W_\alpha(x)\,dx - Q_n(W_\alpha; f^{(m)}) \right|^{1/n} < 1$;

(ii) $\sigma < \frac{1}{2}\tau(\alpha) \Rightarrow \limsup_{n\to\infty} \left| f^{(m)}(\xi) - L_n(W_\alpha; f^{(m)}; \xi) \right|^{1/n} < 1$; *and*

(iii) $\sigma < \tau(\alpha) \Rightarrow \limsup_{n\to\infty} \left| f^{(m)}(\xi) - H_n(W_\alpha; f^{(m)}; \xi) \right|^{1/n} < 1$.

Moreover, (ii) and (iii) hold uniformly on compact subsets of $\mathscr{C}$.

In this paper we generalize the results of Theorem A by considering a more general class of weight functions. In Section 2, we introduce the necessary definitions in order to arrive at the general weight to be considered, and after that we state the main results. In Section 3, we give a sketch of the proof of the main results.

2. Definitions and Results

In order to state our main results, we shall need to define a few classes of weight functions on $\mathfrak{R}$. These classes appeared in [4, Thm. 3.1 and 3.4]

DEFINITION 2.1. Let $W(x) = \exp(-Q(x))$, where $Q(x)$ is even and continuous on $\mathfrak{R}$, with $Q'(x) > 0$ for $x > 0$, $Q''(x)$ exists for $x > 0$, and $Q'''(x)$ exists for large enough x. Furthermore, for some constants $C_1, C_2, C_3 > 0$,

$$C_1 \leq 1 + x|Q''(x)|/Q'(x) \leq C_2 \qquad \text{for } x > 0,$$

and

$$x^2|Q'''(x)|/Q'(x) \leq C_3 \qquad \text{for } x \text{ large enough.}$$

Then we write $W \in VSF$.

Note that if $\alpha > 0$ and $\beta \in \mathfrak{R}$, then $W(x) = \exp(-|x|^{\alpha}(\log(2 + x^2))^{\beta}) \in VSF$. Associated with the weight $W \in VSF$, and whenever it is uniquely defined, $a_n = a_n(W)$ is the Mhaskar–Rahmanov–Saff (MRS) quantity that is defined to be the positive root of the equation

$$n = \frac{2}{\pi}\int_0^1 a_n x Q'(a_n x)(1 - x^2)^{-1/2}\, dx.$$

In the special case $W(x) = W_{\alpha}(x) = \exp(-|x|^{\alpha})$, $\alpha > 0$, $a_n(W)$ takes the simple form [6]:

$$a_n(W_{\alpha}) = \beta_{\alpha} n^{1/\alpha}, \qquad n = 1, 2, 3, \ldots$$

DEFINITION 2.2. Let $w(x) = \prod_{j=1}^{N}|x - z_j|^{\delta_j}$, $x \in \mathfrak{R}$, where $N \geq 1$; $z_1, z_2, \ldots, z_N$ are distinct complex numbers, $\delta_1, \delta_2, \ldots, \delta_N \in \mathfrak{R}$, and if z_j is real, the corresponding $\delta_j > -1/2$. Then w is called a generalized Jacobi factor.

DEFINITION 2.3. Let $\hat{W}(x) = w(x)h(x)W(x)$, where

(i) $W \in VSF$,

(ii) w is a generalized Jacobi factor,

(iii) h is a non-negative measurable function such that $\lim_{|x| \to \infty} h(x) = 1$, and for all c large enough.

$$\int_{-1}^{1} \log h(cx)(1 - x^2)^{-1/2} \, dx > -\infty.$$

Then we write $\hat{W} \in LS$.

Observe that if $\rho > -1$ and $\alpha > 0$, then $\hat{W}(x) = |x|^{\rho/2} \exp(-|x|^{\alpha}) \in LS$.

We now formulate our main results.

THEOREM B. *Let $\hat{W}^2(x)$ be an even weight function on $\mathfrak{R}$, where $\hat{W} \in LS$ such that for some $\alpha > 0$, there exists a constant C_α that depends on α at most, with*

$$\limsup_{n \to \infty} \frac{a_n(W)}{n^{1/\alpha}} \leq C_\alpha;$$

and let

$$\mu(\alpha) = \frac{(1 - a)^{((\alpha+2)/2)}}{2a(\alpha + 2)C_\alpha^\alpha}.$$

Then for all $m = 0, 1, 2, \ldots,$ and $\xi \in \mathscr{C}$, we have

(i) $\sigma < \mu(\alpha) \Rightarrow \limsup_{n \to \infty} \left| \int_R f^{(m)}(x) \hat{W}^2(x) \, dx - Q_n(\hat{W}^2; f^{(m)}) \right|^{1/n} < 1$;

(ii) $\sigma < \frac{1}{2}\mu(\alpha) \Rightarrow \limsup_{n \to \infty} \left| f^{(m)}(\xi) - L_n(\hat{W}^2; f^{(m)}; \xi) \right|^{1/n} < 1$; *and*

(iii) $\sigma < \mu(\alpha) \Rightarrow \limsup_{n \to \infty} \left| f^{(m)}(\xi) - H_n(\hat{W}^2; f^{(m)}; \xi) \right|^{1/n} < 1$.

Moreover, (ii) *and* (iii) *hold uniformly on compact subsets of $\mathscr{C}$.*

At this point we would like to point out that the assumptions of Theorem B will be satisfied if $\hat{W}(x) = W_\alpha(x) = \exp(-|x|^\alpha)$, $\alpha > 0$. In this case C_α can be replaced by β_α, and consequently, we can easily see that Theorem A is a corollary of Theorem B.

3. PROOF OF THEOREM B

The preliminaries and the steps of the proof of Theorem B are essentially the same as those used in the proof of Theorem A. The main difference here comes from taking into consideration the validity of the

Freud's conjecture [5, Th. 2.4]

$$\lim_{n\to\infty} \frac{\gamma_{n-1}}{a_n\gamma_n} = \frac{1}{2}$$

and the additional assumption

$$\limsup_{n\to\infty} \frac{a_n}{n^{1/\alpha}} \le C_\alpha,$$

for the weight W, instead of taking

$$\lim_{n\to\infty}\left(n^{-1/\alpha}\frac{\gamma_{n-1}}{\gamma_n}\right) = \frac{1}{2}$$

for the weight $W_\alpha(x) = \exp(-|x|^\alpha)$, $\alpha > 0$.

So, in light of this brief introduction, we feel that it is appropriate to state without proof the necessary preliminaries and to sketch a proof of Theorem B. We also refer the interested reader to [1] and [4] for more details and references on the material of this section.

Throughout the rest of this paper, we will assume that the assumptions of Theorem B are satisfied. We will also denote $P_n(\hat{W}^2; x)$ by $P_n(x)$, its leading coefficient by γ_n, and its zeros by $x_{1n} > x_{2n} > \cdots > x_{nn}$. Here is the statement of our preliminary results:

LEMMA C.

(i) $$\max_{1\le k\le n-1} \frac{\gamma_{k-1}}{\gamma_k} \le x_{1n} \le 2 \max_{1\le k\le n-1} \frac{\gamma_{k-1}}{\gamma_k}$$

(ii) $$\sum_{k=1}^{[n/2]} (x_{kn})^2 = \sum_{k=1}^{n-1}\left(\frac{\gamma_{k-1}}{\gamma_k}\right)^2$$

(iii) $$\limsup_{R\to\infty} \frac{M_{f^{(m)}}(R)}{2R^\alpha} \le \sigma, \qquad m = 1,2,3,\ldots, \tag{3.1}$$

(iv) for all $z \in \mathscr{C}$, with $|z| \le x_{1n}$, $|P_n(z)| \le 2^{n/2}\gamma_n(x_{1n})^n$

(v) $$\lim_{n\to\infty} \frac{\gamma_{n-1}}{a_n\gamma_n} = \frac{1}{2}$$

(vi) for any $\eta > 0$, there exists $N_\eta \in N$, such that, for all $n \geq N_\eta$, we have

$$x_{1,n+1} \leq 2(\tfrac{1}{2} + \eta) C_\alpha n^{1/\alpha}, \tag{3.2}$$

$$\sum_{k=1}^{[n/2]} (x_{kn})^2 \leq K + \left(\frac{\alpha}{\alpha+2}\right)\left(\frac{1}{2} + \eta\right)^2 C_\alpha^2 n^{((\alpha+2)/\alpha)},$$

$$\frac{1}{\gamma_n^2} \leq A\left(\frac{1}{2} + \eta\right)^{2n} C_\alpha^{2n} (n!)^{2/\alpha}, \tag{3.3}$$

and for all $z \in \mathscr{C}$, $|z| > x_{1,n+1}$,

$$\begin{aligned} |P_n(z) P_{n+1}(z)|^{-1} & \\ \leq \frac{1}{\gamma_n \gamma_{n+1}} \cdot \frac{1}{|z|^{2n+1}} & \\ \cdot \exp\left\{ \frac{2K + 2(\alpha/(\alpha+2))(\frac{1}{2}+\eta)^2 C_\alpha^2 (n+1)^{((\alpha+2)/\alpha)}}{|z|^2 - (x_{1,n+1})^2} \right\}, & \end{aligned} \tag{3.4}$$

where K and A are constants depending on α and η.

For the proofs of (i)–(iv) and (vi), see [1]. As to (v), it is a consequence of the Szegő asymptotic [4, Thm. 3.4]

$$\lim_{n \to \infty} \left\{ \gamma_n \left(\frac{a_n}{2}\right)^{n+(1/2)+\Delta} G[W(a_n x)] \right\} = (2\pi)^{-1/2},$$

where

$$G[W(a_n x)] = \exp\left(\frac{1}{\pi} \int_{-1}^{1} [\log W(a_n x)] (1 - x^2)^{-1/2} \, dx \right),$$

and

$$\Delta = \sum_{j=1}^{N} \delta_j.$$

Outline of the Proof of Theorem B

In view of (3.1), it can be seen that it is sufficient to proof the theorem for $m = 0$ only. From (1.1)–(1.3), it can also be argued that the three parts (i)–(iii) of Theorem B can be proved by, more or less, using the same techniques and estimates. Hence, we proceed at this point to outline the proof of part (i) only.

Let

$$I_n = \frac{\gamma_{n+1}}{\gamma_n} \cdot \frac{1}{2\pi i} \oint_{C_n} \frac{f(z)\,dz}{P_n(z)P_{n+1}(z)},$$

and C_n to be the circle $|z| = R_n$ such that

$$R_n^2 \geq \frac{(x_{1,n+1})^2}{1-\varepsilon}, \qquad \text{for } a < \varepsilon < 1. \tag{3.5}$$

Using (3.4) with $|z| = R_n$, we conclude that

$$\begin{aligned} &|P_n(z)P_{n+1}(z)|^{-1} \\ &\quad \leq \frac{1}{\gamma_n\gamma_{n+1}} \cdot \frac{1}{R_n^{2n+1}} \\ &\qquad \cdot \exp\left\{\frac{2K + 2(\alpha/(\alpha+2))(\frac{1}{2}+\eta)^2 C_\alpha^2 (n+1)^{((\alpha+2)/\alpha)}}{\varepsilon R_n^2}\right\} \end{aligned} \tag{3.6}$$

From the assumption

$$\sigma = \limsup_{R\to\infty} \frac{\log M_f(R)}{2R^\alpha},$$

we can find, for any $\delta > 0$, an $N_\delta \in N$ such that

$$M_f(R_n) \leq \exp\{2(\sigma+\delta)R_n^\alpha\}, \qquad \text{for all } R_n \geq N_\delta. \tag{3.7}$$

Using (3.6), (3.3), and (3.7), we conclude, for large R_n, that

$$\begin{aligned} |I_n| &\leq A\left(\frac{1}{2}+\eta\right)^{2n} C_\alpha^{2n}(n!)^{2/\alpha} \cdot \frac{1}{R_n^{2n}} \\ &\quad \cdot \exp\left\{2(\sigma+\delta)R_n^\alpha \right. \\ &\qquad \left. + \frac{2K + 2(\alpha/(\alpha+2))(\frac{1}{2}+\eta)^2 C_\alpha^2 (n+1)^{((\alpha+2)/\alpha)}}{\varepsilon R_n^2}\right\}. \end{aligned}$$

Next, we choose R_n which will minimize the right-hand side of the last inequality, and which will, at the same time, satisfy (3.5). So, we consider the function

$$h(R) = \frac{1}{R^{2n}} \exp\left\{2(\sigma + \delta)R^{\alpha} + \frac{2K + 2(\alpha/(\alpha+2))(\frac{1}{2} + \eta)^2 C_\alpha^2 (n+1)^{((\alpha+2)/\alpha)}}{\varepsilon R^2}\right\}.$$

By differentiating $h(R)$ and setting $h'(R) = 0$, we get

$$2\alpha(\sigma + \delta)R^{\alpha+2} - \frac{2}{\varepsilon}\left\{2K + 2\left(\frac{\alpha}{\alpha+2}\right)\left(\frac{1}{2} + \eta\right)^2 C_\alpha^2 (n+1)^{((\alpha+2)/\alpha)}\right\} - 2nR^2 = 0. \tag{3.8}$$

Hence, we now choose R_n to satisfy (3.8) and we combine this with (3.2) to obtain

$$R_n^2 \ge \left(\frac{1}{2\varepsilon(\alpha+2)(\sigma+\delta)(1+2\eta)^{\alpha} C_\alpha^{\alpha}}\right)^{(2/(\alpha+2))} (x_{1,n+1})^2.$$

So, (3.5) will be satisfied if

$$\sigma + \delta = \frac{(1-\varepsilon)^{((\alpha+2)/2)}}{2\varepsilon(\alpha+2)C_\alpha^{\alpha}} < \frac{(1-a)^{((\alpha+2)/2)}}{2a(\alpha+2)C_\alpha^{\alpha}} = \mu(\alpha). \tag{3.9}$$

From (3.8), we conclude that

$$2(\sigma+\delta)R_n^{\alpha} = \frac{2n}{\alpha} + \frac{4K + 4(\alpha/(\alpha+2))(\frac{1}{2}+\eta)^2 C_\alpha^2 (n+1)^{((\alpha+2)/\alpha)}}{\varepsilon \alpha R_n^2}.$$

Therefore,

$$|I_n| \le A(\tfrac{1}{2} + \eta)^{2n} C_\alpha^{2n} (n!)^{2/\alpha} \cdot \frac{1}{R_n^{2n}} \cdot \exp\left\{\frac{2n}{\alpha} + \frac{2K + 2(\alpha/(\alpha+2))(\frac{1}{2}+\eta)^2 C_\alpha^2 (n+1)^{((\alpha+2)/\alpha)}}{\varepsilon R_n^2} \cdot \left(\frac{\alpha+2}{\alpha}\right)\right\}.$$

Using (3.8) one more time, we get

$$\frac{1}{R_n^2} \le \left(\frac{2\varepsilon(\alpha + 2)(\sigma + \delta)}{C_\alpha^2} \right)^{(2/(\alpha+2))} n^{-2/\alpha}.$$

By combining the last two inequalities, and then using the Stirling formula and (3.9), we conclude that

$$|I_n| \le Bn^{1/\alpha}\left\{ \left(\frac{1-\varepsilon}{4} \right)(1 + T_1(\eta))\exp\left[t_n + \left(\frac{1-\varepsilon}{2\varepsilon} \right)(1 + T_2(\eta)) \right] \right\}^n,$$

where B is a constant depending on α and η, $t_n \to 0$ as $n \to \infty$, and $T_i(\eta) \to 0$ as $\eta \to 0$, $i = 1, 2$.

Since $g(\varepsilon) = ((1 - \varepsilon)/4)\exp((1 - \varepsilon)/2\varepsilon)$ is continuous, decreasing function on $(0, 1)$ and $g(a) = 1$, it follows that $0 < g(\varepsilon) < 1$ for $a < \varepsilon < 1$. Consequently, we can find, for small η and large n, a number $\rho < 1$, such that

$$\left(\frac{1-\varepsilon}{4} \right)(1 + T_1(\eta))\exp\left\{ t_n + \left(\frac{1-\varepsilon}{2\varepsilon} \right)(1 + T_2(\eta)) \right\} < \rho < 1,$$

which establishes that the series $\sum_k |I_k|$ is convergent.

Therefore, for sufficiently large n,

$$\begin{aligned} &\left| \int_R f(x)\hat{W}^2(x)\,dx - Q_n(\hat{W}^2; f) \right| \\ &\quad \le \sum_{k=n}^{\infty} |I_k| \\ &\quad \le Cn^{1/\alpha}\left\{ \left(\frac{1-\varepsilon}{4} \right)(1 + T_1(\eta))\exp\left[t_n + \left(\frac{1-\varepsilon}{2\varepsilon} \right)(1 + T_2(\eta)) \right] \right\}^n, \end{aligned}$$

where C is a constant depending on α, η, and ε.

Hence, for a small enough η,

$$\limsup_{n \to \infty} \left| \int_R f(x)\hat{W}^2(x)\,dx - Q_n(\hat{W}^2; f) \right|^{1/n} \le g(\varepsilon) < 1,$$

which completes the proof.

Acknowledgment

The authors thank the referee for bringing [4] to their attention, and for rewriting Definitions 2.1 and 2.3 in their current and more general forms.

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